

# Erdős-Szekeres Tableaux

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## Abstract

We explore a question related to the celebrated Erdős-Szekeres Theorem and develop a geometric approach to answer it. Our main object of study is the *Erdős-Szekeres tableau*, or EST, of a number sequence. An EST is the sequence of integral points whose coordinates record the length of the longest increasing and longest decreasing subsequence ending at each element of the sequence. We define the *Order Poset* of an EST in order to answer the question: *What information about the sequence can be recovered by its EST?*

**Keywords:** Monotone Subsequence, Erdős-Szekeres Tableau, Total and Partial Order

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## 1. Introduction

There is a well-known result of Erdős and Szekeres [3] that states:

**Theorem 1** (Erdős-Szekeres). *Given a sequence of  $n$  distinct real numbers, if  $n > rs$ , there exists a monotonically decreasing subsequence of length at least  $r + 1$  or a monotonically increasing subsequence of length at least  $s + 1$ .*

The original appears in the context of a geometric problem: *Can one find a function  $N(n)$  such that any set containing  $N(n)$  distinct points in the plane has a subset of  $n$  points that form a convex polygon?* Thus, Theorem 1 may be considered a work of *combinatorial geometry*, roughly defined as “using combinatorics to solve a problem of geometry.” We would like to contribute to the reverse direction of study, exploring what geometry may tell us about certain problems in combinatorics. For recent work related to the Erdős-Szekeres theorem, see [5, 4, 2].

### 1.1. Sequences and Monotone Subsequences

We provide Seidenberg’s proof of Theorem 1 in order to introduce the main geometric construct with which we will be concerned. See [7] for alternate proofs.

*Theorem 1* (Seidenberg [6])

*Proof.* Let  $A = (a_1, a_2, \dots, a_n)$  be a sequence of distinct real numbers. To each  $a_i$  assign a pair of numbers  $(a_i^+, a_i^-)$ , where  $a_i^+$ , *resp.*  $a_i^-$ , is the length of the longest increasing, *resp.* decreasing, subsequence in  $A$  that ends at  $a_i$ . We first show that for  $i < j$ , the pairs  $(a_i^+, a_i^-)$  and  $(a_j^+, a_j^-)$  must be distinct. Indeed, if  $a_i < a_j$ , then  $a_j^+ \geq a_i^+ + 1$ , while if  $a_i > a_j$ , then  $a_j^- \geq a_i^- + 1$ . This implies that there are  $n$  distinct pairs. Now suppose that  $n > rs$ , for some natural numbers  $r$  and  $s$ . If it were the case that all decreasing

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subsequences had length at most  $r$  and all increasing subsequences had length at most  $s$ , then at most  $rs$  of the pairs  $(a_i^+, a_i^-)$  could be distinct. Therefore, there is at least one index  $i$  such that either  $a_i^- \geq r + 1$  or  $a_i^+ \geq s + 1$ .  $\square$

### 1.2. The Erdős-Szekeres Tableau

The geometric object of study in this paper is the sequence of the points  $\{(a_i^+, a_i^-)\}_{i \leq n}$  defined in the proof of Theorem 1. Henceforth, the term *point* will always refer to a first-quadrant lattice point  $(x, y) \in (\mathbb{Z}^+)^2$ .

**Definition 2.** Given a sequence  $A = (a_1, \dots, a_n)$  of distinct numbers, the **Erdős-Szekeres tableau** (or, **EST**) of  $A$  is the sequence  $T(A) = (t_i)_{i \leq n}$  where  $t_i = (a_i^+, a_i^-)$  and  $a_i^+$ , *resp.*  $a_i^-$ , is the length of the longest increasing, *resp.* decreasing, subsequence in  $A$  that ends at  $a_i$ . An EST may be visualized as the set of points  $\{(a_i^+, a_i^-) \mid 1 \leq i \leq n\}$  together with arrows or number labels that indicate their order in the sequence. Given an EST  $T = (t_i)_{i \leq n}$  and a number  $m \leq n$ , the subsequence  $(t_i)_{i \leq m}$  is called a sub-EST of  $T$  and denoted  $T_m$ .

The points of an EST must be distinct, as discussed in the proof of Theorem 1.

*Example 3.* Let  $A = (1, 3, 8, 5, 7, 4, 6, 2)$ . Then  $T(A) = ((1, 1), (2, 1), (3, 1), (3, 2), (4, 2), (3, 3), (4, 3), (2, 4))$  (see Fig. 1).

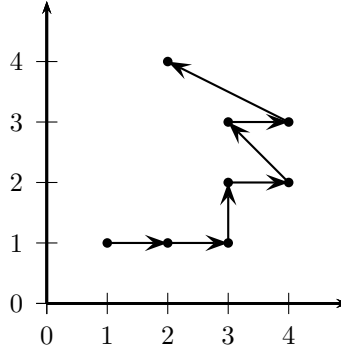


Figure 1: EST for  $A = (1, 3, 8, 5, 7, 4, 6, 2)$ .

**Definition 4.** The set of all ESTs of size  $n$  will be denoted  $\mathcal{T}_n$ .

As evidenced by Example 3, the structure of a typical EST is interesting and non-trivial. In what follows, we will give a concrete description of which sequences of points constitute a valid EST, and conversely, which number sequences  $A = (a_i)_{i \leq n}$  can produce a given EST. The correspondence  $A \rightarrow T(A)$  is not one-to-one. For example, both  $A = (2, 1, 4, 3)$  and  $B = (3, 1, 4, 2)$  have the same EST. Our primary goal is to determine a necessary and sufficient set of relations among the elements of  $A$  such that  $T(A)$  is determined and to classify all sequences  $B$  such that  $T(B) = T(A)$ . A secondary goal is to understand the size of  $\mathcal{T}_n$ .

Certain structures in an EST prove to be important indicators of which sequences of points are, in fact, ESTs. The following terms will apply to any sequence of points, whether it corresponds to an EST or not.

**Definition 5.** Let  $T = (t_i)_{i \leq n} = ((x_i, y_i))_{i \leq n}$  be a sequence of distinct points.

1. The **column above**  $t_i$  is defined by  $\text{Col}(t_i) = \text{Col}(x_i, y_i) = \{(x_i, y) \mid y > y_i\}$ .
2. The **row beyond**  $t_i$  is defined by  $\text{Row}(t_i) = \text{Row}(x_i, y_i) = \{(x, y_i) \mid x > x_i\}$ .
3. The **shadow** of  $T$  is the set of points,  $S(T) = \bigcup_{i \leq n} (\{(x, y_i) \mid x \leq x_i\} \cup \{(x_i, y) \mid y \leq y_i\})$ .
4. The **wall** of  $T$  is the set of points,  $W(T) = (S(T) + \{(1, 0), (0, 1)\}) \setminus S(T)$ , if  $|T| > 0$ , and  $\{(1, 1)\}$  if  $|T| = 0$ .

5. An **edge point** of  $T$  is a member of the set,  $E(T) = \{t \in T \mid \text{Col}(t) \cap T = \emptyset \vee \text{Row}(t) \cap T = \emptyset\}$ .
6. A **corner point** of  $T$  is a member of the set,

$$C(T) = \{(x, y) \in E(T) \mid (x + 1, y) \notin S(T) \wedge (x, y + 1) \notin S(T)\}.$$

When applied to  $T_i$ , we use the convenient notation:  $S_i = S(T_i)$ ,  $W_i = W(T_i)$ , etc.

*Example 6.* Consider the Erdős-Szekeres tableau  $T = T(A)$  of Example 3. The sets  $S(T)$ ,  $W(T)$  and  $C(T)$  are depicted in Fig. 2. The shadow is represented by the light gray region (although only the points with positive integer coordinates within that region are part of the shadow), the wall by the dark gray regions (again, only the integral points), the edge points by large dots, and the corner points (which are also edge points) by large open dots.

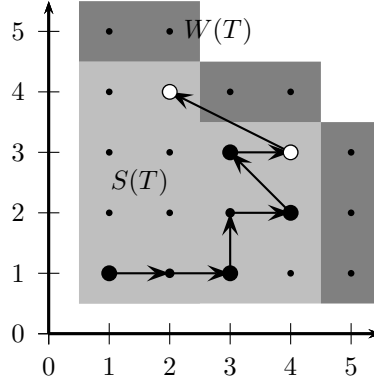


Figure 2:  $S(T)$ ,  $W(T)$ ,  $E(T)$  and  $C(T)$  for  $T = T((1, 3, 8, 5, 7, 4, 6, 2))$ .

The following results follow from the definitions and will become useful in later sections. The proofs are left as an exercises for the reader.

**Lemma 7.** For any  $T \in \mathcal{T}_n$ ,

- (a)  $T_1 \subset T_2 \subset \dots \subset T_n$  and  $S_1 \subset S_2 \subset \dots \subset S_n$
- (b)  $C_n \subseteq E_n \subseteq E_{n-1} \cup \{t_n\} \subseteq T_n \subseteq S_n$
- (c)  $S_n \cap W_n = \emptyset$ .
- (d)  $|W_n| = |E_n| + 1$ .

Furthermore, there is a natural duality relationship concerning sequences and their ESTs. Let  $A = (a_i)_{i \leq n}$  be a sequence of real numbers, and define the dual of  $A$  by  $A^* = (-a_i)_{i \leq n}$ . If we specialize to sequences in  $\{1, 2, \dots, n\}$ , then we may define  $A^* = (n + 1 - a_i)_{i \leq n}$ . This is the same type of duality as Myers uses in [5].

**Definition 8.** Given  $T = T(A)$ , the **dual** EST of  $T$  is  $T^* = T(A^*)$ .

It should be clear that  $T^*$  is obtained by reflecting the points of  $T$  across the diagonal line  $y = x$ , thus rows are exchanged with columns and vice versa.

### 1.3. The Order Poset

In this note, we do not concern ourselves with the actual numeric values of  $a_i$  in a sequence  $A = (a_i)_{i \leq n}$ . Indeed, we only are concerned with the relative order of the  $a_i$ . That is, we regard a sequence  $A$  as a linear order on the set of indices,  $([n], <_A)$ , in which  $i <_A j$  if and only if  $a_i < a_j$  (here,  $[n] = \{1, 2, \dots, n\}$ ). It is natural to ask what order information is preserved in  $T(A)$ . We define an equivalence relation on the set of sequences (linear orders of  $[n]$ ) by  $\{(A, B) \mid T(A) = T(B)\}$ . For  $T \in \mathcal{T}_n$ , define  $[T] = \{A : T(A) = T\}$ . Thus, to answer what information about  $A$  is preserved in  $T(A)$ , we should examine the order relations common to all  $B \in [T(A)]$ .

**Definition 9.** Let  $T \in \mathcal{T}_n$ . Define a partial order  $P(T) = ([n], <_{P(T)})$  by  $i <_{P(T)} j$  if and only if  $i <_A j$  for all sequences  $A \in [T]$ . The poset  $P(T)$  is called the **order poset** of  $T$ .

Certain relations in  $P(T)$  are immediate from the configuration of points of  $T$ . It may come as no surprise that rows and columns of an EST provide information about the order of the sequence that generated the EST.

**Lemma 10** (Row and Column Relations). *Let  $A = (a_i)_{i \leq n}$  be any sequence and  $T(A) = (t_i)_{i \leq n}$ .*

- (a) (Row Relation) *If  $t_j \in \text{Row}(t_i)$ , then  $a_i < a_j$ .*
- (b) (Column Relation) *If  $t_j \in \text{Col}(t_i)$ , then  $a_i > a_j$ .*

*Proof.* We need only prove the row relation, as the column relation would follow by duality in  $T^*$ . Suppose that  $a_i > a_j$ . Since  $t_j \in \text{Row}(t_i)$ , we have  $a_i^+ < a_j^+$  and  $a_i^- = a_j^-$ . Now if  $a_j$  precedes  $a_i$  (that is,  $j < i$ ), then by appending  $a_i$  to the longest increasing subsequence ending at  $a_j$ , we find  $a_i^+ > a_j^+$ , a contradiction. On this other hand, if  $a_i$  precedes  $a_j$  (that is,  $i < j$ ), we may append  $a_j$  to the longest decreasing subsequence ending at  $a_i$ , which would imply  $a_j^- > a_i^-$ , another contradiction. Thus,  $a_i < a_j$  is forced.  $\square$

The following result generalizes the row and column relations. Because these relations may involve a turn around an empty column or row, they are given the suggestive names *hook* and *slice*.

**Lemma 11** (Hook and Slice Relations). *Let  $A = (a_i)_{i \leq n}$  be any sequence and  $T = T(A)$ . Write  $T = (t_i) = ((x_i, y_i))$ .*

- (a) (Hook Relation) *Suppose  $i < j$ . If there is a number  $x$  such that  $x_i \leq x \leq x_j$  and  $\text{Col}(x, y_i) \cap T_j = \emptyset$ , then  $a_i < a_j$ .*
- (b) (Slice Relation) *Suppose  $i < j$ . If there is a number  $y$  such that  $y_i \leq y \leq y_j$  and  $\text{Row}(x_i, y) \cap T_j = \emptyset$ , then  $a_i > a_j$ .*

*Proof.* Suppose  $i < j$  and we have a point  $(x, a_i^-)$  such that  $a_i^+ \leq x \leq a_j^+$  and  $\text{Col}(x, a_i^-) \cap T_j = \emptyset$ . Suppose that  $a_i > a_j$ . Let  $(a_{k_1}, a_{k_2}, \dots, a_j)$  be an increasing subsequence ending at  $a_j$  of length  $a_j^+$ . Since  $x \leq a_j^+$ , we have  $k_x \leq j$ . Suppose that  $a_{k_x}$  preceded  $a_i$ . Then, since  $a_{k_x} \leq a_j < a_i$ , we have  $a_i^+ > a_{k_x}^+ = x$ , a contradiction. Thus  $a_i$  must precede  $a_{k_x}$ . But then  $a_{k_x}^- > a_i^-$ , which puts  $a_{k_x} \in \text{Col}(x, a_i^-) \cap T_j$ , another contradiction. Hence the statement  $a_i > a_j$  is false. The slice relation follows from the hook relation in  $T^*$  by duality.  $\square$

For example, in Fig. 1,  $t_2$  hooks  $t_4$  since  $\text{Col}(2, 1) \cap T_4 = \emptyset$  (the point  $t_8 = (2, 4)$  belongs to  $T_8$ , not  $T_4$ , so the column is considered empty in  $T_4$ ). Therefore,  $b_2 < b_4$  for *any* sequence  $B$  that generates this EST. Also,  $t_4$  slices  $t_6$  since  $\text{Row}(3, 3) \cap T_6 = \emptyset$ . This implies  $b_4 > b_6$  for any  $B$ . Moreover,  $t_4$  hooks  $t_5$  since  $\text{Col}(3, 3) \cap T_5 = \emptyset$ , implying  $b_4 < b_5$ , which illustrates that the hooks and slices do in fact generalize the row (and column) relations.

**Corollary 12.** *Let  $T \in \mathcal{T}_n$ , and  $P = P(T)$  be its order poset. If  $t_i$  hooks  $t_j$ , then  $i <_P j$ . If  $t_i$  slices  $t_j$ , then  $i >_P j$ .*

## 2. Statement of the Main Theorems

In this section we state some theorems regarding the structure of ESTs. Proofs are deferred to Sections 3–5. The first theorem classifies the configuration of ESTs, and answers the question, *Under what conditions is a sequence of points an EST?*

**Theorem 13** (EST Structure Theorem). *A sequence of points,  $T = (t_i)_{i \leq n}$ , is an EST if and only if for each  $1 \leq i \leq n$ ,  $t_i \in W(T_{i-1})$ .*

Now suppose  $T \in \mathcal{T}_n$ , and we find its order poset  $P = P(T)$ . Consider a linear extension  $A$  of  $P$ , and consider its EST,  $T' = T(A)$ . A priori, it may not be the case that  $T = T'$ , however the following theorem will prove that this is indeed the case. For a poset  $P = (X, <_P)$ , let  $L(P)$  be the set of all linear orders on the set  $X$  that extend  $<_P$ .

**Theorem 14** (Linear Extension Theorem). *For any  $T \in \mathcal{T}_n$ ,  $[T] = L(P(T))$ .*

Next we strengthen Cor. 12 by showing that the hook and slice relations are sufficient to generate  $P(T)$ .

**Theorem 15.** *Given  $T \in \mathcal{T}$ , the order poset  $P(T)$  is equal to the transitive closure of the hook and slice relations determined by  $T$ .*

Finally, towards understanding the number of ESTs of size  $n$ , we find asymptotic bounds on  $|\mathcal{T}_n|$ .

**Theorem 16.**  $|\mathcal{T}_n| = n^{n(1-o(1))}$ .

### 3. EST Structure

In this section, we examine the geometric structure of an EST, leading to a proof of Theorem 13. In what follows we assume  $A = (a_i)_{i \leq n}$  is a sequence of distinct numbers, and  $T = T(A) = (t_i)_{i \leq n} = ((a_i^+, a_i^-))_{i \leq n}$  is its EST. The first geometric result, which we call No-Backwards-Placement simply states that higher-index points cannot be in the shadow of lower-index points of the EST.

**Lemma 17** (No-Backward-Placement). *If  $i < j$ , then  $t_j \notin S_i$ .*

*Proof.* If  $t_j \in S_i$ , then there exists  $t_k \in T_i$  such that either  $(a_j^+ < a_k^+) \wedge (a_j^- = a_k^-)$ , or  $(a_j^- < a_k^-) \wedge (a_j^+ = a_k^+)$ . It is sufficient to consider only the first case, as duality would take care of the second. Since  $t_j$  is to the left of  $t_k$  in the same row, we have  $a_j < a_k$ . However, considering the subsequence obtained by appending  $a_j$  to the longest decreasing subsequence ending at  $a_k$ , we find that  $a_j^- \geq a_k^- + 1$ , a contradiction.  $\square$

We now prove Theorem 13.

*Proof.* The *only if* direction is proven by cases. The case  $n = 0$  is clear by definitions. It remains to show that for  $n \geq 1$  we have  $t_n \in W_{n-1}$ . By No-Backward-Placement,  $t_n \notin S_{n-1}$ , so it suffices to show either  $\text{Row}(a_n^+ - 1, a_n^- - 1) \cap T_n \neq \emptyset$  or  $\text{Col}(a_n^+ - 1, a_n^- - 1) \cap T_n \neq \emptyset$ . It is clear that if  $a_n < a_i$  for all  $i < n$ , then  $t_n$  must occupy the extreme upper left point of  $W_{n-1}$ . Dually, if  $a_n > a_i$  for all  $i < n$ , then  $t_n \in W_{n-1}$  at the extreme lower right point. Now suppose that  $a_n^+ > 1$  and  $a_n^- > 1$ . This implies there is some  $i < n$  such that  $a_n^+ = a_i^+ + 1$  (take  $a_i$  to be the penultimate element of the longest increasing subsequence ending at  $a_n$ ), and  $j$  such that  $a_n^- = a_j^- + 1$  (take  $a_j$  analogously in the longest decreasing subsequence ending at  $a_n$ ). Suppose  $\text{Col}(a_i^+, a_j^-) \cap T_n = \emptyset$ . Then it must be the case that  $a_j^+ \geq a_n^+$ , since otherwise  $a_j < a_n$  by Lemma 11, a contradiction. Thus  $t_j \in \text{Row}(a_i^+, a_j^-) \cap T_n$ . Analogously, if we assumed that  $\text{Row}(a_i^+, a_j^-) \cap T_n = \emptyset$ , then  $\text{Col}(a_i^+, a_j^-) \cap T_n \neq \emptyset$  is forced. In either case,  $t_n \in W_{n-1}$ .

For the *if* direction, use induction on  $n$ . Suppose  $T_{n-1}$  is the EST of the sequence  $A' = (a_i)_{i < n}$ , and let  $t \in W_{n-1}$ . We wish to show that there is a number  $a_n$  such that the  $n^{\text{th}}$  point of  $T(A)$  is  $t$ , where  $A = (a_i)_{i \leq n}$ . If  $t$  is in the extreme upper left of  $W_{n-1}$ , then choose  $a_n < a_i$  for all  $i < n$ ; dually, if  $t$  is in the extreme lower right, then choose  $a_n > a_i$  for all  $i < n$ . Now if  $a_n$  is any number, and  $a_n$  is increased slightly so that it swaps position relative to only one value  $a_i$ , then  $a_n^+$  either remains the same or increases by one, and  $a_n^-$  either remains the same or decreases by one (each case depending on whether  $a_i$  was included in the longer monotone subsequence). Thus, the point  $t_n$  of the EST would either remain unchanged, move to the right one unit, move down one unit, or move diagonally down and right one unit each. We have shown above that  $t_n$  must always be in the wall of  $T_{n-1}$ , therefore as the value of  $a_n$  varies from lesser than all  $a_i$  to greater than all  $a_i$ , every point of  $W_{n-1}$  must be visited. Thus, there exists a value  $a_n$  so that the given point  $t$  is the  $n^{\text{th}}$  point of the EST  $T(A)$ .  $\square$

## 4. Order Poset Structure

### 4.1. The Edge Chain

Let  $A = (a_i)_{i \leq n}$  be a sequence of distinct numbers. In order to prove Theorem 14, we develop an inductive method of constructing both the EST  $T(A)$  and its order poset  $P(T(A))$ . Along the way, we find that the two structures are in one-to-one correspondence.

**Lemma 18.** *The placement of points in  $T = T_n$  defines a total order  $<_{T_n}$  on the set of points  $E_{n-1} \cup \{t_n\}$  such that  $t_i <_{T_n} t_j \Rightarrow i <_{P(T)} j$ .*

*Proof.* The proof is by induction. We may assume a total order  $<_{T_{n-1}}$  on  $E_{n-2} \cup \{t_{n-1}\}$ . By Lemma 7(b),  $E_{n-1} \subseteq E_{n-2} \cup \{t_{n-1}\}$  so  $E_{n-1}$  inherits the total order  $<_{T_{n-1}}$ . As part of the inductive hypothesis, assume that the ordering of  $E_{n-1}$  is the clockwise order of points. By “clockwise order”, we mean that we follow the path of wall points starting from upper left to lower right, listing the edge points corresponding to each wall point, and in the event that two edge points correspond to a single wall point (*i.e.*, the wall point at which the wall changes direction from down to right), then follow the rule, *upper-left before lower-right*. As an example, the clockwise order of edge points in Fig. 1 is  $(1, 1), (2, 4), (3, 3), (4, 3), (4, 2), (3, 1)$ . Let  $A$  be any sequence such that  $T_n = T(A)$ , and write  $A = A' \cup \{a_n\}$ . Observe  $T_{n-1} = T(A')$ . By Theorem 13,  $t_n = (a_n^+, a_n^-) \in W_{n-1}$ . We extend the relation  $(E_{n-1}, <_{T_{n-1}})$  to a relation  $(E_{n-1} \cup \{t_n\}, <_{T_n})$  based on the position of  $t_n$  in the wall. There are five cases to consider:

1. If  $t_n$  occupies the extreme upper left point of  $W_{n-1}$ , then the column relation shows  $a_n < a_i$ , where  $t_i$  is the least element of  $E_{n-1}$ . Thus  $t_n$  is the least element of  $(E_{n-1} \cup \{t_n\}, <_{T_n})$ .
2. Dually, if  $t_n$  occupies the extreme lower right point of  $W_{n-1}$ , then  $t_n$  is the greatest element of  $(E_{n-1} \cup \{t_n\}, <_{T_n})$ .
3. If  $a_n^+ > 1$  and  $\text{Col}(a_n^+ - 1, a_n^- - 1) \cap T_n = \emptyset$ , then consider the edge points  $t_i, t_j$  such that  $a_i^+ = a_n^+ - 1$  and  $a_j^+ = a_n^+$ . Clearly  $t_i$  and  $t_j$  are consecutive in the order  $<_{T_{n-1}}$ . Then  $t_i$  hooks  $t_n$ , implying  $a_i < a_n$ , while  $a_j > a_n$  due to the column relation. Extend  $<_{T_{n-1}}$  by inserting  $t_n$  between  $t_i$  and  $t_j$ .
4. Dually, if  $a_n^- > 1$  and  $\text{Row}(a_n^+ - 1, a_n^- - 1) \cap T_n = \emptyset$ , then we find consecutive  $t_i, t_j$  such that  $a_i^- = a_n^-$  and  $a_j^- = a_n^- - 1$ . Extend the linear order by inserting  $t_n$  between  $t_i$  and  $t_j$ .
5. If none of the above conditions apply, then there exists  $t_i \in E_{n-1}$  in the same row as  $t_n$  (to the left), and  $t_j \in E_{n-1}$  in the same column (below). As conditions 3 and 4 do not apply we must have  $a_j^+ = a_i^+ + 1$  and  $a_j^- = a_i^- - 1$ . Since  $a_i < a_n < a_j$  is forced, and  $t_i$  and  $t_j$  are consecutive, the linear order is extended by inserting  $t_n$  between  $t_i$  and  $t_j$ .

Note that in all cases the new set of edge points,  $E_n \subseteq E_{n-1} \cup \{t_n\}$ , satisfies the condition that  $<_{T_n}$  is the clockwise order.  $\square$

**Corollary 19.** *Within the Order Poset  $P(T)$ , there are chains  $\mathcal{E}_i$  corresponding to the total orders  $(E_i, <_{T_i})$  of Lemma 18, for each  $i \leq n$ . Call  $\mathcal{E}_n$  the **edge chain** of  $T$ .*

### 4.2. The Order Poset Theorem

**Lemma 20.** *Let  $T \in \mathcal{T}_n$ . The set of order relations  $\{<_{T_i}\}_{i \leq n}$  from Lemma 18 is necessary and sufficient for a sequence  $A$  to generate  $T$ .*

*Proof.* Necessity is clear from Lemma 18. To show sufficiency, we must show that  $T$  can be reconstructed from the relations. By induction, it suffices to show that  $t_n$  can be uniquely determined, given  $T_{n-1}$  and the order relation  $<_{T_n}$ . By Lemma 7 and Theorem 13, there are exactly  $|E_{n-1}| + 1$  distinct points at which  $t_n$  can be placed, and each placement induces a distinct linear ordering on  $E_n \subseteq E_{n-1} \cup \{t_n\}$ . Thus the placement of  $t_n$  is determined by  $(E_{n-1} \cup \{t_n\}, <_{T_n})$ .  $\square$

**Corollary 21.** *The set of hook and slice relations are sufficient to generate  $P(T)$ .*

*Proof.* By Lemma 20,  $P(T)$  is equal to the transitive closure of the relations  $\{<_{T_i}\}_{i \leq n}$  determined by  $T$ . All of these relations are either hooks or slices.  $\square$

**Corollary 22.** *Let  $T = T_n \in \mathcal{T}_n$ . A sequence  $A$  of the numbers  $\{1, 2, \dots, n\}$  generates  $T$  if and only if  $(A, <)$  is a linear extension of  $([n], <_{P(T)})$ . A real number sequence  $A$  generates  $T$  if and only if there is an order-preserving map of sequences  $A \rightarrow A'$  such that  $A'$  is a linear extension of  $([n], <_{P(T)})$ .*

Cor. 21 implies Theorem 15, while Cor. 22 implies Theorem 14 .

#### 4.3. Long chains within the order poset

With a view towards counting  $\mathcal{T}_n$ , we find a chain in  $P = P(T)$  (for arbitrary  $T \in \mathcal{T}_n$ ) which includes the edge chain  $\mathcal{E}_n$  as a subchain. Write the corner points in linear order:  $C(T) = \{t_{r_1}, t_{r_2}, \dots, t_{r_k}\}$ . Call two corner points,  $t_{r_i}$  and  $t_{r_{i+1}}$ , **consecutive**.

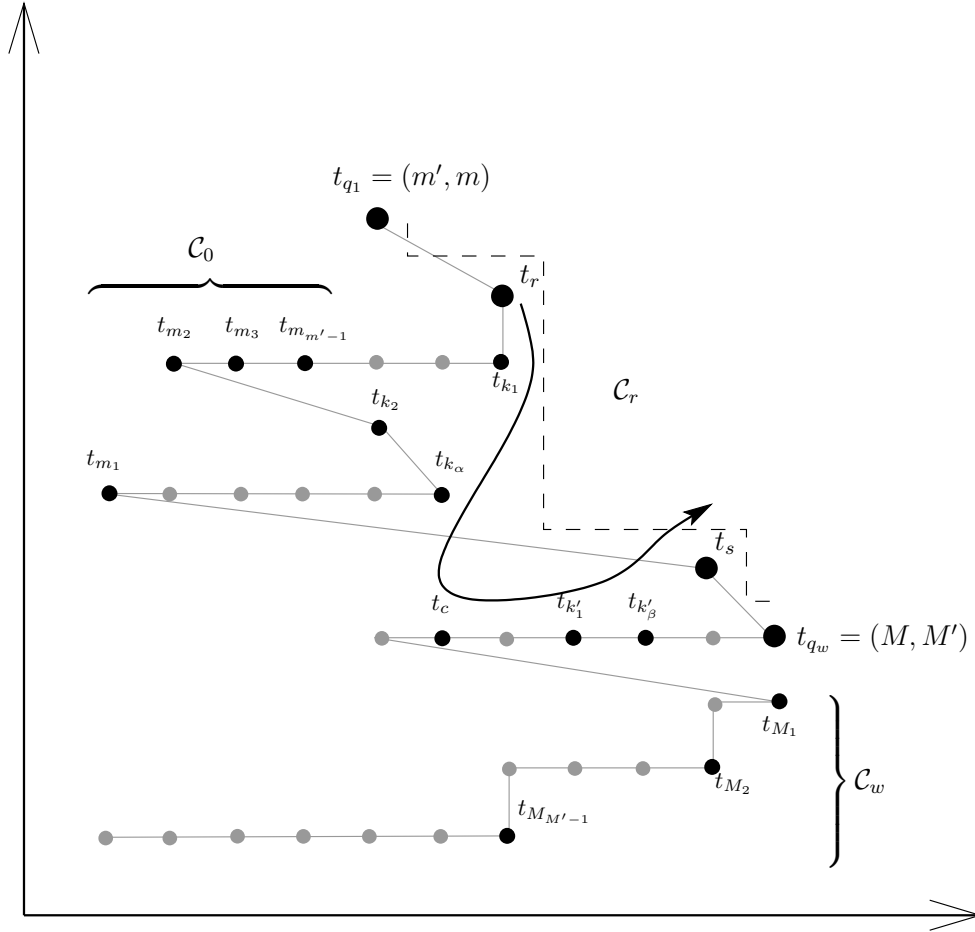


Figure 3: Chains in  $P(T)$ . The diagram shows an example EST, with points labeled as in the proof of Lemma 23. Corner points are shown as larger dots. The curved arrow indicates the direction of points in the chain  $\mathcal{C}_r$ .

**Lemma 23.** *Let  $t_r = (x_r, y_r)$  and  $t_s = (x_s, y_s)$  be consecutive corner points of  $T$ . There is a chain from  $r$  to  $s$  in  $P$  with at least  $(y_r - y_s) + (x_s - x_r) - 1$  intermediate points.*

*Proof.* Let  $\mathcal{C}_r$  be the portion of the edge chain strictly between  $t_r$  and  $t_s$ . Note that  $|\mathcal{C}_r| = \alpha + \beta$ , where  $\alpha = x_s - x_r - 1$  and  $\beta = y_r - y_s - 1$ . In terms of the poset  $P$ , we may write:

$$\mathcal{C}_r = k_1 <_P k_2 <_P \dots <_P k_\alpha <_P k'_1 <_P k'_2 <_P \dots <_P k'_\beta,$$

where an index  $k_i$  refers to an edge point  $t_{k_i}$  corresponding to a vertical portion of the wall, while an index  $k'_j$  refers to an edge point  $t_{k'_j}$  corresponding to a horizontal portion of the wall. It may be helpful to refer to Fig. 3. We shall find an additional point  $t_c$  such that  $k_\alpha <_P c <_P k'_1$ . Without loss of generality, assume  $k_\alpha < k'_1$ , since the case  $k_\alpha > k'_1$  can be handled in  $T^*$ . Set  $t_c = (x_c, y_{k'_1})$  where  $x_c = \max\{x \mid (x, y_{k'_1}) \in T_{k_\alpha}\}$ . Such a point  $t_c$  exists by the EST Structure Theorem, and No-Backwards-Placement implies  $x_c < x_{k'_1}$ . Now  $c <_P k'_1$  by the horizontal relation, and  $c >_P k_\alpha$  because  $\text{Row}(t_c) \cap T_{k_\alpha} = \emptyset$  implies that  $t_c$  slices  $t_{k_\alpha}$ .  $\square$

**Lemma 24.** *The poset  $P(T)$  has height at least  $M + m - 1$ , where*

$$M = M(T) = \max\{x \mid (x, y) \in T\} \quad \text{and} \quad m = m(T) = \max\{y \mid (x, y) \in T\}. \quad (1)$$

*Proof.* Let  $\mathcal{L}$  be the chain obtained from the edge chain by inserting each internal point  $t_c$  as in Lemma 23. There are exactly  $|C_n| - 1$  such internal points. A simple counting argument gives the relation  $|E_n| = M + m - |C_n|$ , and the result follows from  $|\mathcal{L}| = |E_n| + |C_n| - 1$ .  $\square$

**Proposition 25.** *For all  $T \in \mathcal{T}_n$ ,  $\text{Height}(P(T)) \geq 2\sqrt{n} - 1$ .*

*Proof.* Let  $M$  and  $m$  be as in Eqn. (1). Since the points of  $T$  must fall in the region  $[1, M] \times [1, m]$  we have  $Mm \geq n$ . Minimizing  $M + m$  over this (integer) constraint yields  $M + m \geq \lceil \sqrt{n} \rceil + \lfloor \sqrt{n} \rfloor$  which is well approximated by  $2\sqrt{n}$ . The result follows from Lemma 24.  $\square$

## 5. Quantitative Results

Our main goal in this section is to find bounds on the order of  $\mathcal{T}_n$ , but we will also discuss the distribution of the sizes of the equivalence classes  $[T] = L(P(T))$ . We will close with some explicit values for small  $n$ .

First observe that Prop. 25 can be used to get a crude lower bound on the number of linear extensions of a given poset  $P = P(T)$  and, by extension,  $|\mathcal{T}_n|$ . Suppose  $P$  has height  $h = 2\lceil \sqrt{n} \rceil - 1$  (recall all such posets have minimum and maximum elements). Among posets (of this type) the number of linear extensions decreases with the number of additional relations and so to maximize the size of the equivalence class,  $\Delta_n = \max\{|[T]| \mid T \in \mathcal{T}_n\}$ , the  $(n - h)$  points outside the unique maximal chain must form an antichain. To extend  $P$  to a linear order there are  $(n - h)!$  ways to order the antichain (this produces a poset consisting of 2 internally disjoint chains) and then  $\binom{n-2}{h-2}$  ways of interlacing the chains. Hence  $\Delta_n \leq (n - h)! \binom{n-2}{h-2}$  from which one could get an explicit lower bound  $|\mathcal{T}_n| \geq \frac{n!}{\Delta_n}$ . However, this bound is very weak as the distribution of class sizes is skewed towards the low end (see Fig. 4). We give a stronger bound in Theorem 16 which shows that the order of  $\mathcal{T}_n$  grows exponentially. The following is a proof of Theorem 16.

*Proof.* The upper bound is immediate as  $|\mathcal{T}_n| \leq n!$ . For the lower bound consider the tableau

$$\{(1, 1), (1, 2), (1, 3) \dots (1, k)\}.$$

Observe that any extension has a wall of size at least  $k$  as the number of columns cannot be decreased. Sequentially choosing the remaining  $n - k$  points we get that  $|\mathcal{T}_n| \geq k^{n-k}$  for all  $k \leq n$ . Taking  $k = \frac{n}{\log n}$  gives the desired result.  $\square$

*Remark 26.* It may be possible to derive a more explicit lower bound on the number of EST's by taking into account the distribution of sizes of equivalence classes in  $\mathcal{T}_n$  for arbitrary  $n$ . Unfortunately, we do not know much about the distribution except that it resembles a power law (with negative exponent). Even knowing the number of ESTs that have unique realizations (equivalently, that have linear posets) would be helpful. We present some data in Fig. 5.

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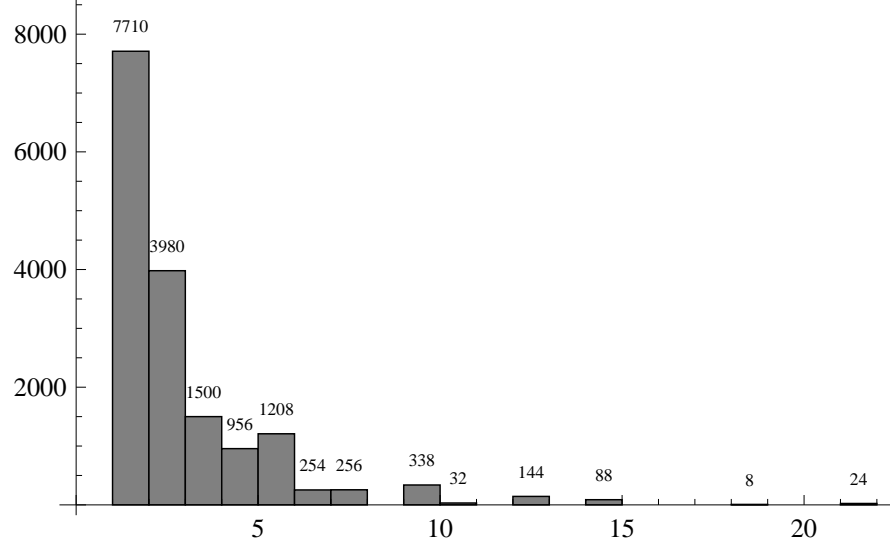


Figure 4: Histogram: Frequency of  $|[T]|$  for  $T \in \mathcal{T}_8$

$n$	$ \mathcal{T}_n $	$\Delta_n$	$U_n$
$\leq 3$	$n!$	1	1
4	22	2	.83
5	96	3	.78
6	480	5	.68
7	2682	10	.57
8	16498	21	.47
9	110378	44	.44
10	795582		
11	6131722		
12	50224736		
13	434989688		

Figure 5: Values of  $|\mathcal{T}_n|$ ,  $\Delta_n = \max\{|[T]| \mid T \in \mathcal{T}_n\}$ , and  $U_n = \frac{1}{|\mathcal{T}_n|}|\{T \in \mathcal{T}_n \mid |[T]| = 1\}|$ .

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